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ON THE RADIUS OF CONVEXITY AND BOUNDARY DISTORTION OF SCHLICHT FUNCTIONS(1)

BY
DAVID E. TEPPER

Abstract. Let $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular and univalent for $|z|<1$ and map $|z|<1$ onto a region which is starlike with respect to $w=0$. If r_0 denotes the radius of convexity of $w=f(z)$, $d_0=\min |f(z)|$ for $|z|=r_0$, and $d^*=\inf |\beta|$ for $f(z)\neq\beta$, then it has been conjectured that $d_0/d^*\geqq 2/3$. It is shown here that $d_0/d^*\geqq 0.343\ldots$, which improves the old estimate $d_0/d^*\geqq 0.268\ldots$. In addition, sharp estimates for r_0 are given which depend on the value of $|a_2|$.

1. **Introduction.** It is shown in [2] that if $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is regular and univalent for $|z|<1$, then there is a positive number r_0, such that $w=f(z)$ maps $|z|\leqq r_0$ onto a convex region. Furthermore, it is shown that $r_0\geqq 2-\sqrt{3}$ for all functions $w=f(z)$ which are regular and univalent for $|z|<1$. From this we see that associated with every function $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ regular and univalent for $|z|<1$, there is a radius of convexity r_0 which is the largest number such that $w=f(z)$ maps $|z|\leqq r_0$ onto a convex region and need not map $|z|\leqq r$ onto a convex region when $r>r_0$.

In this paper the following question is considered: Let $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular and univalent for $|z|<1$ and map $|z|<1$ onto a region which is starlike with respect to $w=0$. If r_0 denotes the radius of convexity of $w=f(z)$, $d_0=\min_{|z|=r_0} |f(z)|$, and $d^*=\inf |\beta|, f(z)\neq\beta$, then in [9] it is conjectured that $d_0/d^*\geqq 2/3$. This lower limit cannot be improved since it is attained for the function $w=f(z)=z(1-z)^{-2}$. In this paper the conjecture is demonstrated for certain classes of functions, while for other functions lower estimates for d_0/d^* are found. Presently, the best estimate for all starlike maps is $d_0/d^*>2-\sqrt{3}=.268\ldots$, see [9].

In order to obtain estimates for d_0/d^*, we study how the second coefficient in the expansion of the function $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ regular and univalent for $|z|<1$ affects certain properties of this function. This type of problem was first studied by Gronwall in [4]. In this paper we generalize results of Finkelstein [1] to the class of

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functions $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which are starlike of order α. These functions, which are characterized by $\operatorname{Re}(zf'(z)/f(z)) \geqq \alpha$ for $0 \leqq \alpha \leqq 1$, were first introduced in [8]. Furthermore we give sharp lower bounds for the radius of convexity which depend on the second coefficient in the expansion of $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which is regular and univalent for $|z|<1$ and maps $|z|<1$ onto a region which is starlike with respect to $w=0$. Using these estimates we show $d_0/d^* \geqq .343\ldots$. The method used to obtain this estimate is then generalized to the class of functions which has p-fold rotational symmetry. It is here that the conjecture for d_0/d^* is demonstrated for certain classes of functions.

NOTATION. Let U denote the class of functions $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which are regular and univalent for $|z|<1$. Let St denote the class of functions $w=f(z) \in U$ which map $|z|<1$ onto a starlike region with respect to $w=0$. Finally, let St_α denote the class of functions $w=f(z) \in U$ which are starlike of order α, for $0 \leqq \alpha \leqq 1$. It is well known that $\mathrm{St}_0=\mathrm{St}$.

2. **Preliminaries.** In this section we prove two lemmas concerning functions which have positive real part.

LEMMA 1. *If* $P(z)=1+bz+\sum_{n=2}^{\infty} b_n z^n$ *is regular and has* $\operatorname{Re} P(z)>0$ for $|z|<1$, *then*

$$\operatorname{Re} P(z) \geqq \frac{1-|z|^2}{1+b|z|+|z|^2} \tag{1}$$

where $b \geqq 0$. *Furthermore, this result is sharp for each value of* b, $0 \leqq b \leqq 2$ *by considering the functions* $P_b(z)=(1-z^2)(1-bz+z^2)^{-1}$.

Proof. Since $\operatorname{Re} P(z)>0$ and $P(0)=1$, the function $P(z)$ is subordinate to the function $(1+z)(1-z)^{-1}$; see [5, p. 228]. Therefore, there exists a function $h(z)$ which is regular for $|z|<1$ with $h(0)=0$ and $|h(z)|<1$ such that:

$$P(z) = \frac{1+h(z)}{1-h(z)} = 1+bz+\sum_{n=2}^{\infty} b_n z^n. \tag{2}$$

A direct computation gives $h(z)=bz/2+\cdots$. Therefore, by a generalized form of Schwarz's Lemma [5, p. 167],

$$|h(z)| \leqq |z| \frac{|z|+b2^{-1}}{1+b2^{-1}|z|} = |z| \frac{2|z|+b}{2+b|z|}. \tag{3}$$

Another direct computation shows

$$\operatorname{Re} P(z) = \frac{1-|h(z)|^2}{|1-h(z)|^2} \geqq \frac{1-|h(z)|}{1+|h(z)|}, \tag{4}$$

since the right-hand side of (4) is monotone decreasing with respect to $|h(z)|$, applying (3) to (4) we obtain

$$\operatorname{Re} P(z) \geqq \frac{1-|z|^2}{1+b|z|+|z|^2}. \tag{5}$$

A direct computation shows sharpness.

LEMMA 2. *If*

$$P(z) = 1+bz+\sum_{n=2}^{\infty} b_n z^n$$

is regular and has $\operatorname{Re} P(z)>0$ *for* $|z|<1$, *then*

$$|P(z)| \leqq \frac{1+b|z|+|z|^2}{1-|z|^2}, \tag{6}$$

where $b \geqq 0$. *Furthermore, this result is sharp for each value of* b, $0 \leqq b \leqq 2$, *by considering the functions* $P^b(z)=(1-z^2)(1-bz+z^2)^{-1}$.

Proof. Consider the function $Q(z)=P(-z)^{-1}$. Since $Q(z)$ obeys the hypothesis of Lemma 1, we have

$$\frac{1}{|P(z)|} = |Q(-z)| \geqq \operatorname{Re} Q(-z) \geqq \frac{1-|z|^2}{1+b|z|+|z|^2}. \tag{7}$$

A direct computation shows sharpness.

3. **Estimates for the class** St_α. In this section we give estimates for the function $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n \in \mathrm{St}_\alpha$. For notational convenience we will write $a_2=a$. It is no loss of generality to suppose $a \geqq 0$. If this is not the case, then consider the function $w=e^{i\theta}f(e^{-i\theta}z)$ where $\theta=\arg a$.

THEOREM 1. *If* $w=f(z) \in \mathrm{St}_\alpha$, *then*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geqq \frac{(1-\alpha)+a\alpha|z|+(1-\alpha)(2\alpha-1)|z|^2}{(1-\alpha)+a|z|+(1-\alpha)|z|^2}. \tag{8}$$

Proof. By the principle of subordination, there exists a function $P(z)=z+bz+\sum_{n=2}^{\infty} b_n z^n$ which is regular and has $\operatorname{Re} P(z)>0$ for $|z|<1$, such that

$$zf'(z)/f(z) = (1-\alpha)P(z)+\alpha; \tag{9}$$

see [5, p. 228]. Furthermore, a direct computation shows

$$zf'(z)/f(z) = 1+az+\cdots. \tag{10}$$

Equating coefficients of z in (9), we have $b=a(1-\alpha)^{-1}$. Therefore, by Lemma 1 we have

$$\operatorname{Re} P(z) \geqq \frac{1-|z|^2}{1-a(1-\alpha)^{-1}|z|+|z|^2}. \tag{11}$$

Using (11) on (9), we obtain (8).

THEOREM 2. *If* $w=f(z) \in \mathrm{St}_\alpha$, *then*

$$|f(z)| \geqq |z| \left[\frac{1-\alpha}{(1-\alpha)+a|z|+(1-\alpha)|z|^2}\right]^{1-\alpha}. \tag{12}$$

Proof. If $z=re^{i\theta}$, then

$$\frac{\partial}{\partial r}\log\left|\frac{f(z)}{z}\right| = \operatorname{Re}\frac{zf'(z)}{f(z)}-1. \tag{13}$$

Applying Theorem 1 to (13), we obtain

$$\frac{\partial}{\partial r}\log\left|\frac{f(z)}{z}\right| \geqq -(1-\alpha)\frac{a+2(1-\alpha)r}{(1-\alpha)+ar+(1-\alpha)r^2}.$$

Integrating from $r=0$ to $r=|z|<1$, after taking exponents, we obtain inequality (12).

THEOREM 3. *If* $w=f(z)\in \mathrm{St}_\alpha$, *then*

$$|f'(z)| \geqq (1-\alpha)^{1-\alpha}\left\{\frac{(1-\alpha)+a\alpha|z|+(2\alpha-1)(1-\alpha)|z|^2}{[(1-\alpha)+a|z|+(1-\alpha)|z|^2]^{2-\alpha}}\right\}. \tag{14}$$

Proof. The result follows by applying inequality (12) to inequality (8).

A direct computation shows that

$$f_{\alpha,a}(z) = z\left[\frac{1-\alpha}{(1-\alpha)-az+(1-\alpha)z^2}\right]^{1-\alpha}$$

gives extremal functions for Theorems 1, 2, and 3. Using Lemma 2 in a similar manner in which Lemma 1 was used, the following theorem may be proven.

THEOREM 4. *If* $w=f(z)\in \mathrm{St}_\alpha$, *then*

$$\left|\frac{zf'(z)}{f(z)}\right| \leqq \frac{1+a|z|+(1-2\alpha)|z|^2}{1-|z|^2}, \tag{15}$$

$$|f(z)| \leqq |z|\left[\frac{1+|z|}{1-|z|}\right]^{a/2}\frac{1}{(1-|z|^2)^{1-\alpha}}, \tag{16}$$

$$|f'(z)| \leqq \left[\frac{1+|z|}{1-|z|}\right]^{a/2}\frac{1+a|z|+(1-2\alpha)|z|^2}{(1-|z|^2)^{2-\alpha}}. \tag{17}$$

A direct computation shows that

$$F_{\alpha,a}(z) = z\left[\frac{1+z}{1-z}\right]^{a/2}\frac{1}{(1-z^2)^{1-\alpha}}$$

gives extremal functions for Theorem 4.

Let K denote the class of functions $w=f(z)\in U$ which map $|z|<1$ onto a convex region. Clearly, $K\subset \mathrm{St}$. In [13], Strohhäcker proved $K\subset \mathrm{St}_{1/2}$. In [10], Schild proved that

$$|z|(1+|z|)^{-1} \leqq |f(z)| \leqq |z|(1-|z|)^{-1} \tag{18}$$

when $w=f(z)\in \mathrm{St}_{1/2}$. Since the extremal function for (18), which is $w=f(z)=z(1-z)^{-1}$ maps $|z|<1$ onto a convex region, we see that the same estimate for

$|f(z)|$ holds for the class K as the class $\mathrm{St}_{1/2}$. However, in [4], Gronwall proved that

$$|f(z)| \geqq \frac{1}{\sqrt{(1-a^2)}} \operatorname{Arctan} \frac{|z|\sqrt{(1-a^2)}}{1+a|z|}, \quad \text{for } 0 \leqq a < 1,$$
$$\geqq \frac{|z|}{1+|z|}, \quad \text{for } a = 1,$$

when $w=f(z) \in K$. It is interesting to notice that

$$\frac{1}{\sqrt{(1-a^2)}} \operatorname{Arctan} \frac{|z|\sqrt{(1-a^2)}}{1+a|z|} > \frac{|z|}{\sqrt{(1+2a|z|+|z|^2)}}$$

when $a \neq 1$.

4. **Radius of convexity estimates.** The following two lemmas enable us to give sharp estimates for r_0 when $w=f(z) \in \mathrm{St}$ has a preassigned second coefficient. Again we will write $a_2=a$ and we will assume $a \geqq 0$.

LEMMA 3. *If* $P(z)=1+bz+\sum_{n=3}^{\infty} b_n z^n$ *is regular and has* $\operatorname{Re} P(z)>0$ *for* $|z|<1$, *then*

$$\left| z\frac{P'(z)}{P(z)} \right| \leqq \frac{|z|}{1-|z|^2} \frac{b|z|^2+4|z|+b}{|z|^2+b|z|+1} \tag{19}$$

where $b \geqq 0$.

Proof. As was shown in Lemma 1, there exists a function $h(z)=bz/2+\cdots$ regular for $|z|<1$ with $|h(z)|<1$, such that $P(z)=(1+h(z))/(1-h(z))$. Furthermore, since $h(0)=0$ and $|h(z)|<1$, there exists a function $\phi(z)$ regular for $|z|<1$, such that $h(z)=z\phi(z)$. Using this we obtain $P(z)=(1+z\phi(z))/(1-z\phi(z))$. Taking the logarithmic derivatives of both sides we have

$$\frac{P'(z)}{P(z)} = 2\frac{\phi(z)+z\phi'(z)}{1-z^2\phi^2(z)}.$$

Using the triangle inequality, we have

$$\left| \frac{P'(z)}{P(z)} \right| \leqq 2\frac{|\phi(z)|+|z|\,|\phi'(z)|}{1-|z|^2|\phi(z)|^2}. \tag{20}$$

In [5], it is proven that

$$|\phi'(z)| \leqq \frac{1-|\phi(z)|^2}{1-|z|^2}. \tag{21}$$

Since the right-hand side of (20) is monotone increasing with respect to $|\phi'(z)|$, by substituting (21) into (20) we obtain

$$\left| \frac{P'(z)}{P(z)} \right| \leqq \frac{2}{1-|z|^2} \left[\frac{|\phi(z)|(1-|z|^2)+|z|(1-|\phi(z)|^2)}{1-|z|^2|\phi(z)|^2} \right]. \tag{22}$$

We wish to show the expression in square brackets in (22) is monotone increasing with respect to $|\phi(z)|$. To do this consider

$$g(x) = \frac{x(1-r^2)+r(1-x^2)}{1-r^2x^2} \tag{23}$$

where $0 \leqq x = |\phi(z)| \leqq 1$ and $0 \leqq r = |z| < 1$. Differentiating (23) we obtain

$$g'(x) = \frac{(1-r^2)(1-rx)^2}{(1-r^2x^2)^2} \geqq 0$$

because $0 \leqq r < 1$ and $0 \leqq x \leqq 1$.

Furthermore, by inequality (3) we have

$$|\phi(z)| = \left|\frac{h(z)}{z}\right| \leqq \frac{2|z|+b}{2+b|z|}.$$

Since the expression in square brackets in (22) is monotone increasing with respect to $|\phi(z)|$, we obtain

$$\left|\frac{P'(z)}{P(z)}\right| \leqq \frac{2}{1-|z|^2}\frac{\left(\frac{2|z|+b}{2+b|z|}\right)(1-|z|^2)+|z|\left[1-\left(\frac{2|z|+b}{2+b|z|}\right)^2\right]}{1-|z|^2\left(\frac{2|z|+b}{2+b|z|}\right)^2}$$

$$= \frac{1}{1-|z|^2}\frac{(1-|z|^2)(b|z|^2+4|z|+b)}{(1-|z|^2)(|z|^2+b|z|+1)}$$

which completes the proof.

LEMMA 4. *If* $w=f(z) \in \mathrm{St}$, *then*

$$\mathrm{Re}\,\frac{zf''(z)}{f'(z)}+1 \geqq \frac{1}{1-|z|^2}\frac{1-a|z|-6|z|^2-a|z|^3+|z|^4}{1+a|z|+|z|^2}. \tag{24}$$

Proof. If

$$P(z) = zf'(z)/f'(z) = 1+az+\cdots,$$

then since $w=f(z) \in \mathrm{St}$, Theorem 1 gives

$$\mathrm{Re}\,P(z) \geqq \frac{1-|z|^2}{1+a|z|+|z|^2}. \tag{25}$$

Direct computation gives

$$zf''(z)/f'(z)+1 = P(z)+zP'(z)/P(z).$$

Therefore, applying (25) and (19) to this equation we obtain

$$\mathrm{Re}\,\frac{zf''(z)}{f'(z)}+1 \geqq \mathrm{Re}\,P(z)-\left|z\frac{P'(z)}{P(z)}\right|$$

$$\geqq \frac{1}{1-|z|^2}\frac{1-a|z|-6|z|^2-a|z|^3+|z|^4}{1+a|z|+|z|^2}$$

which completes the proof of Lemma 4.

We are now ready to give estimates for the radius of convexity for functions in St.

THEOREM 5. *If $w=f(z)\in \mathrm{St}$ and has the radius convexity r_0, then*

$$r_0 \geqq r_0(a) = \frac{a+\sqrt{(a^2+32)}-\sqrt{[2a^2+2a\sqrt{(a^2+32)}+16]}}{4}. \tag{26}$$

This estimate is sharp for each a, $0\leqq a\leqq 2$ by considering the functions $f_a(z)=z/(1-az+z^2)$.

Proof. By [2] we have that $w=f(z)\in U$ maps $|z|\leqq r$ onto a convex region if and only if $\operatorname{Re}(zf''(z)/f'(z))+1\geqq 0$ for $|z|\leqq r$. Therefore by Lemma 4, $w=f(z)$ will map $|z|\leqq r$ onto a convex region if

$$\operatorname{Re}\frac{zf''(z)}{f'(z)}+1 \geqq \frac{1}{1-|z|^2}\frac{1-a|z|-6|z|^2-a|z|^3+|z|^4}{1+a|z|+|z|^2} \geqq 0.$$

for $|z|<r$. Therefore, the radius of convexity of $w=f(z)$ is greater than or equal to the least positive root of

$$q_a(r) = 1-ar-6r^2-ar^3+r^4 = 0, \tag{27}$$

which is exactly $r_0(a)$. A direct computation verifies sharpness.

5. **Estimates for d_0/d^*.** In this section we prove $d_0/d^*>.343\ldots$. In order to obtain this estimate, we need the following lemmas.

LEMMA 5. *If $w=f(z)\in \mathrm{St}$, then*

$$\begin{aligned} d^* &\leqq 2/(a+2), & 0\leqq a\leqq 1,\\ &\leqq 2/3a, & 1\leqq a\leqq 2. \end{aligned} \tag{28}$$

Proof. The estimate $d^*\leqq 2/3a$ for $1\leqq a\leqq 2$ is proven by Netanyahu, see [6]. We will show $d^*\leqq 2/(a+2)$ for $0\leqq a\leqq 1$. Let $g(w)=w+c_2w^2+\cdots$ denote the inverse function to $w=f(z)$. A direct computation shows $c_2=-a$. Consider the function

$$h(\zeta) = \frac{1}{d^*}\frac{g(d^*\zeta)}{(1+g(d^*\zeta))^2}.$$

Since the composition of univalent functions is univalent and $h'(0)=1$, $h(\zeta)\in U$. The second coefficient of $h(\zeta)$ is $-d^*(2+a)$. By [2], we have $d^*(2+a)\leqq 2$.

Applying Theorem 7 to Theorem 2 when $\alpha=0$, Lemma 5 gives the following lower bound for d_0/d^*.

LEMMA 6. *If $w=f(w)\in \mathrm{St}$, then*

$$\begin{aligned} d_0/d^* &\geqq \frac{a+2}{2}\left[\frac{r_0(a)}{1+ar_0(a)+r_0^2(a)}\right], & 0\leqq a\leqq 1,\\ &\geqq \frac{3}{2}a\left[\frac{r_0(a)}{1+ar_0(a)+r_0^2(a)}\right], & 1\leqq a\leqq 2. \end{aligned} \tag{29}$$

In order to minimize the right-hand side of (29), the following lemma is needed.

LEMMA 7. *The function*

$$r_0(a) = \frac{a+\sqrt{(a^2+32)}-\sqrt{[2a^2+2a\sqrt{(a^2+32)}+16]}}{4}$$

is monotone decreasing for $0 \leqq a \leqq 2$.

Proof. By (27), $r_0(a)$ satisfies the equation

$$1-ar_0(a)-6r_0^2(a)-ar_0^3(a)+r_0^4(a) = 0.$$

Using implicit differentiation, we have

$$r_0'(a) = \frac{r_0(a)[r_0^2(a)+1]}{4r_0^3(a)-3ar_0^2(a)-12r_0(a)-a}$$

which has a positive numerator. As for the denominator, we have

$$4r_0^3(a)-3ar_0^2(a)-12r_0(a)-a \leqq 4r_0(a)(r_0^2(a)-3) < 0$$

because $0 \leqq a \leqq 2$ and $0 < r_0(a) < 1$. Therefore, $r_0'(a) < 0$ and $r_0(a)$ decreases.

THEOREM 5. *If* $w=f(z) \in \mathrm{St}$, *then* $d_0/d^* \geqq .343\ldots$.

Proof. If $0 \leqq a \leqq 1$, we have

$$d_0/d^* \geqq \frac{a+2}{2}\left[\frac{r_0(a)}{1+ar_0(a)+r_0^2(a)}\right] = \frac{h(a)}{2}.$$

By (27) we have

$$a = \frac{1-6r_0^2(a)+r_0^4(a)}{r_0(a)(1+r_0^2(a))}.$$

Applying this equation to the function $h(a)$, we obtain

$$\begin{aligned} h(a) &= \frac{1}{2}\left[\frac{(1-r_0(a))^2(1+4r_0(a)+r_0^2(a))}{(1-r_0(a))^2(1+r_0(a))^2}\right] \\ &= \frac{1}{2}\left[1+\frac{2r_0(a)}{(1+r_0(a))^2}\right]. \end{aligned}$$

Hence,

$$h'(a) = \left[\frac{1-r_0(a)}{(1+r_0(a))^3}\right]r_0'(a).$$

Therefore, $h'(a) < 0$ because $r_0'(a) < 0$ and $0 < r_0(a) < 1$. From this we obtain $d_0/d^* \geqq h(a)/2 \geqq h(1)/2$ when $0 \leqq a \leqq 1$.

If $1 \leqq a \leqq 2$, we have

$$d_0/d^* \geqq \frac{3}{2}a\left[\frac{r_0(a)}{1+ar_0(a)+r_0^2(a)}\right] = \frac{3}{2}k(a).$$

By (27) we have

$$ar_0(a) = \frac{1-6r_0^2(a)+r_0^4(a)}{(1+r_0^2(a))}.$$

Substituting this equation into the function $k(a)$ we obtain

$$k(a) = \frac{1}{2}\left[\frac{1-6r_0^2(a)+r_0^4(a)}{(1-r_0^2(a))^2}\right].$$

Since

$$k'(a) = -4r_0(a)r_0'(a)\left[\frac{1+r_0^2(a)}{(1-r_0^2(a))^3}\right],$$

$r_0'(a)<0$ gives $k'(a)>0$. From this we obtain

$$d_0/d^* \geqq \tfrac{3}{2}k(a) \geqq \tfrac{3}{2}k(1)$$

when $1 \leqq a \leqq 2$. Thus we obtain the following

$$d_0/d^* \geqq \tfrac{1}{2}h(1) = \tfrac{3}{2}k(1) = .343\ldots.$$

6. **Functions with p-fold rotational symmetry.** Let St_p denote the class of functions $w=f(z)=z+\sum_{n=1}^{\infty} a_{np+1}z^{np+1} \in \mathrm{St}$ which have p-fold rotational symmetry. Using methods similar to those used in the previous sections, the following estimates may be proven for this class of functions where $a=a_{p+1}\geqq 0$,

$$\frac{|z|}{\sqrt[p]{[1+pa|z|^p+|z|^{2p}]}} \leqq |f(z)| \leqq |z|\left[\frac{1+|z|^p}{1-|z|^{2p}}\right]^{pa/2}\left[\frac{1}{1-|z|^{2p}}\right]^{1/p}$$

$$r_0 \geqq \left[\frac{p^2a+\sqrt{[p^4a^2+16(1+p)]}-\sqrt{[2p^4a^2+2p^2a\sqrt{(p^4a^2+16+16p)}+16p]}}{4}\right]^{1/p}$$

$$\begin{aligned} d^* &\leqq \left[\frac{2}{ap+2}\right]^{1/p}, & 0 \leqq a \leqq \frac{1}{p}, \\ &\leqq \left[\frac{2}{3ap}\right]^{1/p}, & \frac{1}{p} \leqq a \leqq \frac{2}{p}. \end{aligned}$$

From these inequalities we obtain

THEOREM 7. *If $w=f(z)\in \mathrm{St}_p$, then $d_0/d^*>2/3$ for $p\geqq 5$.*

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TEMPLE UNIVERSITY,
PHILADELPHIA, PENNSYLVANIA 19122